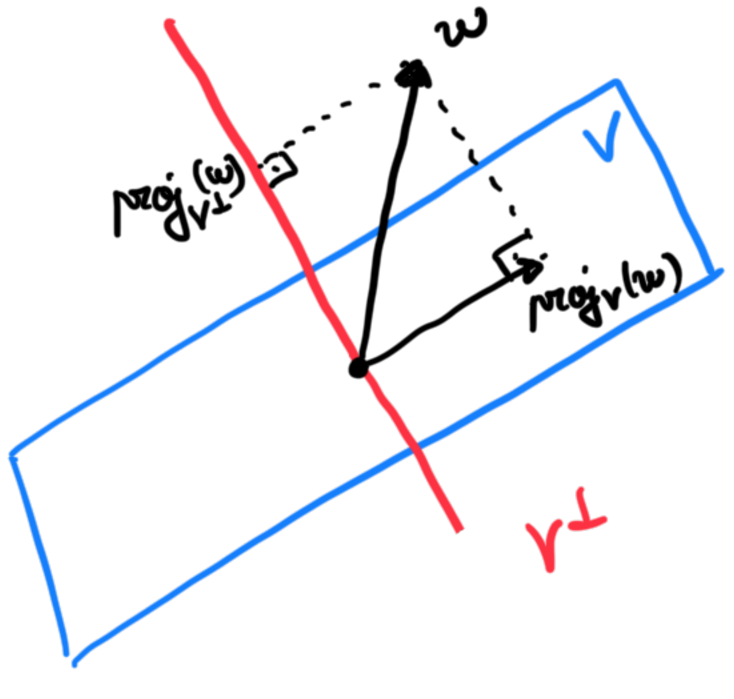
Last time : suppose $v_1, \ldots, v_k$ are mutually orthogonal vectors in $\mathbb{R}^n$, and let $V = \text{span}\{v_1, \ldots, v_k\}$

$$\forall w \in \mathbb{R}^n, \quad \text{proj}_V(w) = \frac{w \cdot v_1}{\|v_1\|^2} v_1 + \ldots + \frac{w \cdot v_k}{\|v_k\|^2} v_k \in V$$

projection of w onto V

$$\forall w \in \mathbb{R}^n, \quad w = \text{proj}_V(w) + \text{proj}_{V^\perp}(w)$$



• also, $\dim V + \dim V^\perp = n$ (complementary dimensions)

rank-nullity theorem

$$\boxed{\begin{aligned} \mathrm{Col}(A)^\perp &= \mathrm{Ker}(A^T) \\ \mathrm{Row}(A)^\perp &= \mathrm{Ker}(A) \end{aligned}}$$ $\forall$ matrix A

- $\{v_1, \dots, v_k\}$ are orthonormal if mutually orthogonal and all have length 1 $\iff$ $\begin{aligned} v_i \cdot v_i &= 1 \\ v_i \cdot v_j &= 0 \end{aligned}$ $\forall\, i \neq j$

$\Updownarrow$

$A^T A = I_k$, where $A = (v_1 | \dots | v_k)$

Today: note that if $A \in \mathbb{R}^{n \times k}$ has orthonormal columns

$\Updownarrow$

$A^T A = I_k$

then $k \leq n$ $\begin{cases} \text{if } k < n, \text{ then } A A^T \neq I_n \\ \text{if } k = n, \text{ then } A A^T = I_n \end{cases}$ $\quad A^T = A^{-1}$

$\underset{\substack{\text{dim } n \\ \text{if } AB = I_n}}{\mathrm{Col}(AB)} \subseteq \underset{\text{dim } k}{\mathrm{Col}(A)}$

(**Recall:** orthogonal matrices = square matrices U whose columns are orthonormal, so $U^T U = U U^T = I_n$

$\Updownarrow$

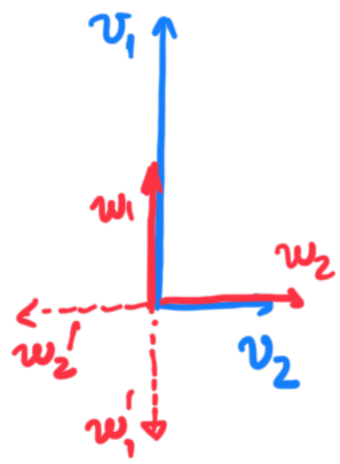
$U^{-1} = U^T$)

$$S = \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \right\} \subseteq \mathbb{R}^4 \qquad \begin{matrix} n=4 \\ k=3 \end{matrix}$$

$$\quad\; \overset{\shortparallel}{v_1} \qquad \overset{\shortparallel}{v_2} \qquad \overset{\shortparallel}{v_3}$$

$$\left. \begin{aligned} v_1 \cdot v_2 &= 0 \\ v_1 \cdot v_3 &= 0 \\ v_2 \cdot v_3 &= 0 \end{aligned} \right\} \Rightarrow S \text{ is orthogonal}$$

$$\left. \begin{aligned} \|v_1\| &= \sqrt{v_1 \cdot v_1} = \sqrt{2} \\ \|v_2\| &= \sqrt{v_2 \cdot v_2} = \sqrt{2} \\ \|v_3\| &= \sqrt{v_3 \cdot v_3} = \sqrt{2} \end{aligned} \right\} \Rightarrow S \text{ is not orthonormal}$$



but $w_1 = \dfrac{v_1}{\pm\sqrt{2}}$, $\quad w_2 = \dfrac{v_2}{\pm\sqrt{2}}$, $\quad w_3 = \dfrac{v_3}{\pm\sqrt{2}}$

then $\{w_1, w_2, w_3\}$ is an orthonormal set

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad A^T = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad A A^T = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \neq I_4$$

Projection: assume $V = \mathrm{span}\{v_1, \ldots, v_k\} \subseteq \mathbb{R}^n$

$\hookrightarrow$ orthonormal

$$\mathrm{proj}_V : \mathbb{R}^n \to \mathbb{R}^n, \qquad \mathrm{proj}_V(w) = \frac{w \cdot v_1}{\cancel{\|v_1\|^2}_{\,1}} v_1 + \cdots + \frac{w \cdot v_k}{\cancel{\|v_k\|^2}_{\,1}} v_k$$

$$= (w \cdot v_1) v_1 + \cdots + (w \cdot v_k) v_k$$

THM 23.1: let V, $v_1, \ldots, v_k$ as above

$\hookrightarrow$ orthonormal

$$A = (v_1 \ \ldots \ v_k), \quad A^T A = I_k$$

then $\boxed{\mathrm{proj}_V(w) = A A^T w}$

$\mathrm{Im}\ \mathrm{proj}_V = V = \mathrm{Col}(A)$

Proof:

$$A A^T w = (v_1 | \cdots | v_k) \underbrace{\begin{pmatrix} v_1^T \\ \hline \vdots \\ \hline v_k^T \end{pmatrix} w}$$

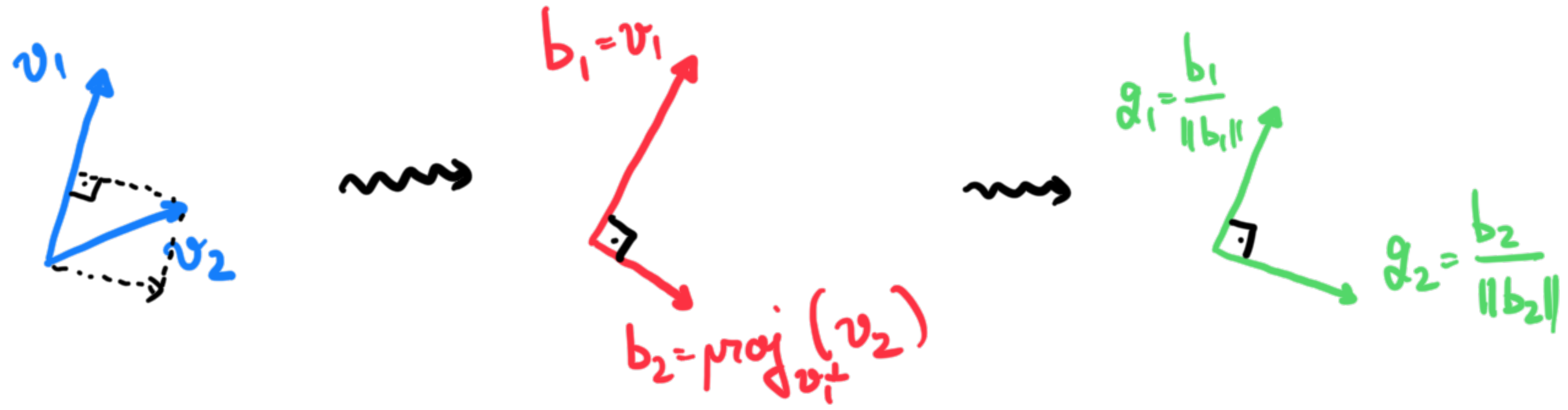
$$= (v_1 | \cdots | v_k) \begin{pmatrix} v_1 \cdot w \\ \vdots \\ v_k \cdot w \end{pmatrix}$$

$$\ldots (v_k \cdot w) = \mathrm{proj}_V(w)$$

$$= v_1 (v_1 \cdot w) + \dots + v_k (v_k \cdot w) = \mathrm{proj}_V(w).$$ $\square$

Gram-Schmidt orthogonalization/orthonormalization

is the process for converting any basis of any subspace $V \subseteq \mathbb{R}^n$ to an orthogonal/orthonormal basis.



ORTHOGONAL ORTHONORMAL

Step 0: pick any basis $v_1, \dots, v_k$ of $V \subseteq \mathbb{R}^n$

Step 1: $b_1 = v_1$

why?

Step 2: $b_2 = v_2 - \mathrm{proj}_{b_1}(v_2)$

$$= v_2 - \frac{v_2 \cdot b_1}{b_1 \cdot b_1} b_1$$

$$b_2 \cdot b_1 = v_2 \cdot b_1 - \frac{v_2 \cdot b_1}{\cancel{b_1 \cdot b_1}} \cancel{b_1 \cdot b_1} = 0$$

Step 3: $b_3 = v_3 - \mathrm{proj}_{\mathrm{span}\{b_1, b_2\}}(v_3)$

$$= v_3 - \frac{v_3 \cdot b_1}{b_1 \cdot b_1} b_1 - \frac{v_3 \cdot b_2}{b_2 \cdot b_2} b_2$$

$$\vdots$$

Step k: $$b_k = v_k - \operatorname{proj}_{\operatorname{span}\{b_1, \dots, b_{k-1}\}}(v_k)$$

$$= v_k - \frac{v_k \cdot b_1}{b_1 \cdot b_1} b_1 - \dots - \frac{v_k \cdot b_{k-1}}{b_{k-1} \cdot b_{k-1}} b_{k-1}$$

THM 23.2

- $\{b_1, \dots, b_k\}$ and $\{v_1, \dots, v_k\}$ are bases of V
- $b_1, \dots, b_k$ are orthogonal, hence they form an orthogonal basis of V.

Proof. $b_1 = v_1$

$$b_2 = v_2 - p_{12} b_1$$

$$p_{ij} := \frac{v_j \cdot b_i}{b_i \cdot b_i} \in \mathbb{R}$$

$$b_3 = v_3 - p_{13} b_1 - p_{23} b_2$$

$$\vdots$$

$$b_k = v_k - p_{1k} b_1 \dots - p_{k-1,k} b_{k-1}$$

$\{v_1, \dots, v_k\}$ and $\{b_1, \dots, b_k\}$ are triangular w.r.t. each other

$$v_k = b_k + b_{k-1} p_{k-1,k} + \cdots + b_1 p_{1k} \quad (*)$$

$\underline{b} = \{b_1, \ldots, b_k\}$

$\underline{v} = \{v_1, \ldots, v_k\}$

$$P_{\underline{b} \leftarrow \underline{v}} = \left([v_1]_{\underline{b}} \; [v_2]_{\underline{b}} \; \cdots \; [v_k]_{\underline{b}} \right) = \begin{pmatrix} 1 & p_{12} & p_{13} & \cdots & p_{1k} \\ 0 & 1 & p_{23} & \cdots & p_{2k} \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & 0 & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

$k \times k$ upper triangular matrix with 1's on diagonal hence invertible; because any linear relation between b's can be transformed (via $P_{\underline{v} \leftarrow \underline{b}}$) into a linear relation between the v's,

then $\{v_1, \ldots, v_k\}$ independent $\Rightarrow$ $\{b_1, \ldots, b_k\}$ independent

$(*)$ implies that $\operatorname{span}\{v_1, \ldots, v_k\} \subseteq \operatorname{span}\{b_1, \ldots, b_k\}$

$\overset{\shortparallel}{V}$

must be an equality, because both b's and v's are independent

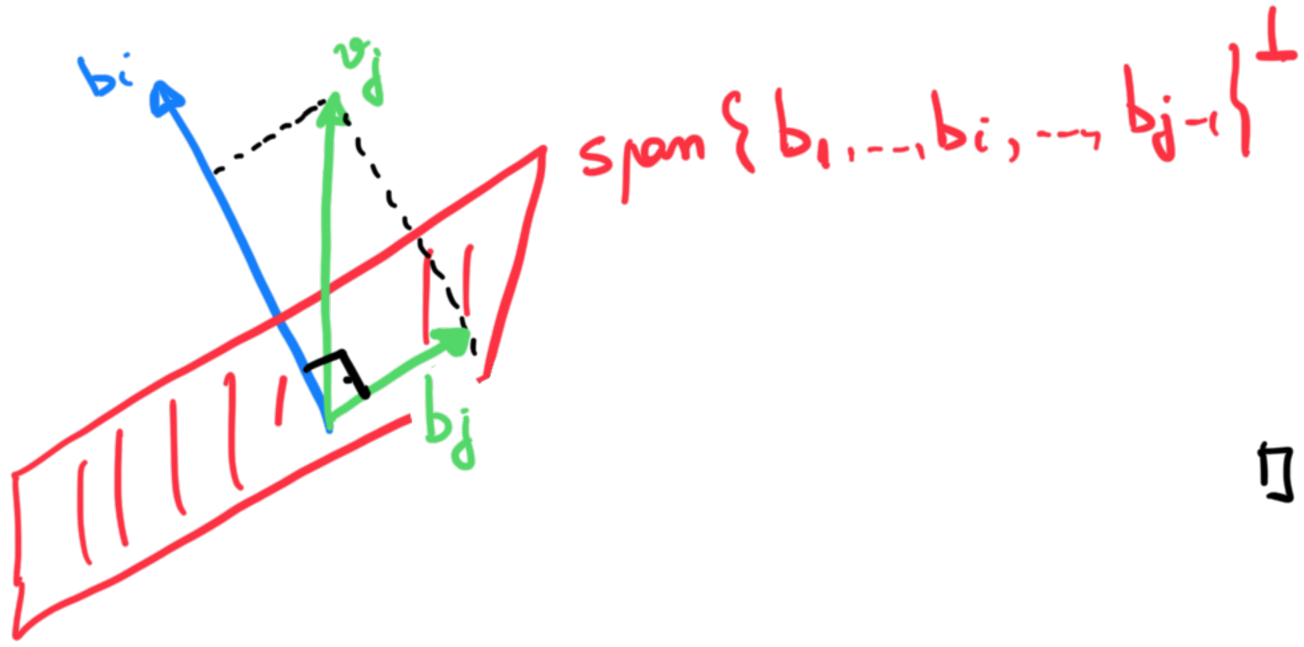
- why are $b_1, \ldots, b_k$ orthogonal? Take any $i < j$; then

$$b_i \cdot b_j = b_i \cdot \underbrace{\left(v_j - \operatorname{proj}_{\operatorname{span}\{b_1,\dots,b_{j-1}\}}(v_j)\right)}_{\parallel \text{ Thm } 22.8}$$

$$b_i \cdot \operatorname{proj}_{\operatorname{span}\{b_1,\dots,b_{j-1}\}^{\perp}}(v_j) = 0 \quad (*)$$

because $b_i \perp \operatorname{span}\{b_1,\dots,b_i,\dots,b_{j-1}\}^{\perp}$

Geometric explanation of (*)



$\square$

Example: find an orthogonal basis for

$$V = \operatorname{span}\left\{ \underset{v_1}{\begin{pmatrix}1\\-1\\1\\0\end{pmatrix}}, \underset{v_2}{\begin{pmatrix}1\\0\\1\\0\end{pmatrix}}, \underset{v_3}{\begin{pmatrix}1\\1\\0\\2\end{pmatrix}} \right\} \subseteq \mathbb{R}^4$$

G-S : $b_1 = v_1 = \begin{pmatrix}1\\-1\\1\\0\end{pmatrix}$

$$b_2 = v_2 - \operatorname{proj}_{\operatorname{span}\{b_1\}}(v_2)$$

$$= v_2 - \frac{v_2 \cdot b_1}{\dots} b_1 = \begin{pmatrix}1\\0\\\dots\end{pmatrix} - \frac{2}{\dots}\begin{pmatrix}1\\-1\\\dots\end{pmatrix} = \begin{pmatrix}1/3\\2/3\\1/3\\\dots\end{pmatrix}$$

$$\cdots \frac{\quad}{b_1 \cdot b_1} \cdots \begin{pmatrix} \vdots \\ 0 \end{pmatrix} \quad 3 \begin{pmatrix} \vdots \\ 0 \end{pmatrix} \quad \begin{pmatrix} \vdots \\ 0 \end{pmatrix}$$

$$b_3 = v_3 - \mathrm{proj}_{\mathrm{span}\{b_1, b_2\}}(v_3)$$

$$= v_3 - \frac{v_3 \cdot b_1}{b_1 \cdot b_1} b_1 - \frac{v_3 \cdot b_2}{b_2 \cdot b_2} b_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix} - \frac{0}{3} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{2/3} \begin{pmatrix} 1/3 \\ 2/3 \\ 1/3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \\ -1/2 \\ 2 \end{pmatrix}$$

Hence $b_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$, $b_2 = \frac{1}{3} \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$, $b_3 = \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 4 \end{pmatrix}$

G-S orthonormalization

$$\begin{matrix} v_1 \\ \vdots \\ v_k \end{matrix} \rightsquigarrow \begin{matrix} b_1 \\ \vdots \\ b_k \end{matrix} \rightsquigarrow \begin{matrix} q_1 \\ \vdots \\ q_k \end{matrix} \qquad \text{where } q_i = \frac{b_i}{\|b_i\|}$$

ORTHOGONAL ORTHONORMAL

Ex: $q_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$, $q_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$, $q_3 = \frac{1}{\sqrt{18}} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 4 \end{pmatrix}$

New topic: QR decomposition/factorization

THM 23.3: assume $A \in \mathbb{R}^{n \times k}$ has linearly independent columns; then there is a unique way to write

$$A = QR$$

$n \times k$ with orthonormal columns

$k \times k$ upper triangular with positive diagonal entries

Q and R come from Gram-Schmidt on the columns of A

Ex: $A = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$, so $v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $v_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$

$b_1 = v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \rightsquigarrow \|b_1\| = \sqrt{14}$

$b_2 = v_2 - \mathrm{proj}_{\mathrm{span}\{b_1\}}(v_1) = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} - \dfrac{\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$= \begin{pmatrix} 4 \\ \ldots \end{pmatrix} - \frac{16}{\ldots} \begin{pmatrix} 1 \\ \ldots \end{pmatrix}$

$$= \frac{1}{7}\begin{pmatrix} 12 \\ 3 \\ -6 \end{pmatrix} = \frac{3}{7}\begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$$

$$\|b_2\| = \frac{3\sqrt{3}}{\sqrt{7}}$$

so G-S gives $g_1 = \dfrac{b_1}{\|b_1\|} = \dfrac{1}{\sqrt{14}}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $g_2 = \dfrac{b_2}{\|b_2\|} = \dfrac{1}{\sqrt{21}}\begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$

$$Q = \begin{pmatrix} g_1 & g_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{14}} & \frac{4}{\sqrt{21}} \\ \frac{2}{\sqrt{14}} & \frac{1}{\sqrt{21}} \\ \frac{3}{\sqrt{14}} & \frac{-2}{\sqrt{21}} \end{pmatrix}$$

what is $R = \begin{pmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{pmatrix} \in \mathbb{R}^{2\times 2}$?

$$\begin{pmatrix} v_1 & v_2 \end{pmatrix} = A = QR = \begin{pmatrix} g_1 & g_2 \end{pmatrix}\begin{pmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{pmatrix}$$

$\Downarrow$

$$v_1 = r_{11} g_1$$

$$v_2 = r_{22} g_2 + r_{12} g_1$$

So $R = P_{\underline{g} \leftarrow \underline{v}}$ is the change of coordinate matrix from the $v_1, \ldots, v_k$ basis to the $q_1, \ldots, q_k$ basis

One can also get R from Gram-Schmidt

$$b_1 = v_1 \qquad\qquad v_1 = b_1$$

$$b_2 = v_2 - \frac{16}{7} b_1 \quad \Longleftrightarrow \quad v_2 = b_2 + \frac{16}{7} b_1$$

$$q_1 = \frac{b_1}{\|b_1\|} \qquad\qquad b_1 = \sqrt{14} \cdot q_1$$

$$q_2 = \frac{b_2}{\|b_2\|} \quad \Longleftrightarrow \quad b_2 = \frac{3\sqrt{3}}{\sqrt{7}} q_2$$

$$\Longrightarrow \quad v_1 = \sqrt{14} \cdot q_1$$

$$v_2 = \frac{3\sqrt{3}}{\sqrt{7}} q_2 + \frac{16\sqrt{2}}{\sqrt{7}} q_1$$

$$\Downarrow$$

$$R = \begin{pmatrix} \sqrt{14} & \frac{16\sqrt{2}}{\sqrt{7}} \\ 0 & \frac{3\sqrt{3}}{\sqrt{7}} \end{pmatrix}$$

Sanity check:

$$QR = \begin{pmatrix} \frac{1}{\sqrt{14}} & \frac{4}{\sqrt{21}} \\ \frac{2}{\sqrt{14}} & \frac{1}{\sqrt{21}} \\ \frac{3}{\sqrt{14}} & \frac{-2}{\sqrt{21}} \end{pmatrix} \begin{pmatrix} \sqrt{14} & \frac{16\sqrt{2}}{\sqrt{7}} \\ 0 & \frac{3\sqrt{3}}{\sqrt{7}} \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} = A$$

General procedure:

$$\begin{matrix} v_1 \\ \vdots \\ v_k \end{matrix} \rightsquigarrow \begin{matrix} b_1 = v_1 \\ \vdots \\ b_k = v_k - p_{1k} b_1 - \dots - p_{k-1,k} b_{k-1} \end{matrix} \rightsquigarrow \begin{matrix} g_1 = \dfrac{b_1}{\|b_1\|} \\ \vdots \\ g_k = \dfrac{b_k}{\|b_k\|} \end{matrix}$$

$$\begin{matrix} v_1 = b_1 \\ \vdots \\ v_k = b_k + p_{k-1,k} b_{k-1} + \dots + p_{1k} b_1 \end{matrix} \qquad \begin{matrix} b_1 = d_1 g_1 \\ \vdots \\ b_k = d_k g_k \end{matrix}$$

where $d_i = \|b_i\|$

Combining these formulas gives us

$$v_1 = d_1 g_1$$

$$v_2 = d_2 g_2 + d_1 p_{12} g_1$$

$$\vdots$$

$$v_k = d_k g_k + d_{k-1} p_{k-1,k} g_{k-1} + \dots + d_1 p_{1k} g_k$$

so $$R = P_{\underline{g} \leftarrow \underline{v}} = \begin{pmatrix} d_1 & d_1 p_{12} & \cdots & d_1 p_{1k} \\ 0 & d_2 & \cdots & d_2 p_{2k} \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & & d_k \\ 0 & 0 & \cdots & \end{pmatrix}$$

$$\begin{pmatrix} d_1 & \\ & \ddots \end{pmatrix} \begin{pmatrix} 1 & p_{12} & \cdots & p_{1k} \\ & 1 & & \end{pmatrix}$$

$$= \begin{pmatrix} d_2 & & 0 \\ & \ddots & \\ 0 & & d_k \end{pmatrix} \begin{pmatrix} 1 & \cdots & r_{2k} \\ & \ddots & \vdots \\ 0 & & 1 \end{pmatrix}$$

$$Q = \begin{pmatrix} q_1 & \cdots & q_k \end{pmatrix}$$

together gives us $A = QR$